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# LARGE ELASTIC DEFORMATIONS OF SHELLS WITH THE INCLUSION OF TRANSVERSE NORMAL STRAIN†

V. BIRICIKOGLU‡ and ARTURS KALNINS§

Lehigh University, Bethlehem, Pennsylvania

**Abstract**—For large, elastic deformations and incompressible materials, the thickness of a shell must change when the shell is being stretched. In this paper, a theory of shells is given which admits a prescribed thickness change on the boundaries of the shell and is capable of predicting a symmetric thickness change throughout the shell. The governing equations are written for an incompressible material with a Mooney-type constitutive law.

## INTRODUCTION

A THEORY of shells which is subjected to the kinematic constraint that the thickness of the shell before and after deformation remains the same is not realistic when large strains are admitted in the deformation process. To enforce such a constraint, the density of the material would have to change in a special way during deformation, and since most materials which are capable of undergoing large strains are nearly incompressible, such a density change cannot be admitted.

A considerable amount of literature can be found on the subject of the removal of the assumption that the thickness remains the same, or, equivalently, that the transverse normal strain is zero, but most of it is aimed at the linear theory. A thorough examination of this topic via the linear theory is included in an NACA report by Hildebrand *et al.* [1], where a shell theory is derived which incorporates quadratic terms with respect to the thickness coordinate in all displacement components. The thickness change is effected through the linear and quadratic terms in the transverse displacement component which is assumed in [1] in the form

$$v_3(x_1, x_2, x_3) = u_3(x_1, x_2)+x_3\beta_3(x_1, x_2)+x_3^2w_2(x_1, x_2) \tag{1}$$

where $(x_1, x_2)$ are the coordinates on the reference surface and $x_3$ is directed along its normal. The linear term, $\beta_3$, represents a thickness change which is symmetric about the middle surface of the shell, while $w_2$ gives rise to an antisymmetric thickness change.

When $\beta_3$ and $w_2$ are included in the theory, then, as shown in [1], additional resultants, $S^\alpha$ and $T^\alpha$, which are the first and second moments of the transverse shear stress, appear in the governing equations. The inclusion of $S^\alpha$ and $T^\alpha$ permits the prescription of additional

† This research was supported by the National Aeronautics and Space Administration Grant NGR 39-007-017.
‡ Assistant Professor of Mechanics.
§ Professor of Mechanics.

boundary conditions which involve $\beta_3$, $S^\alpha$, $w_2$ and $T^\alpha$. Thus, if $S^\alpha$ is retained, then a symmetric thickness change can be prescribed on the boundaries, while the retention of $T^\alpha$ admits the prescription of an antisymmetric thickness change.

The complexity of the final form of the equations of [1] led to the reconsideration of the effects of the transverse shear and normal strains by Reissner [2] and Naghdi [3], again by means of the linear theory. Both [2] and [3] present theories in which $S^\alpha$ and $T^\alpha$ are assumed zero. This simplification, however, is achieved at the expense of not being able to prescribe the thickness change along the edges of the shell.

In the realm of finite deformations, solutions of various shell problems have been obtained by means of the membrane theory for elastic, isotropic, incompressible materials. For example, Rivlin and Thomas [4] analyzed the strain distribution around a hole in a circular sheet, Adkins and Rivlin [5] solved the problem of the inflation of a circular membrane, Kydoniefs and Spencer [6] solved the inflation problem of a circular torus, and Kydoniefs [7] presented solutions to the same problem without any restriction to the cross section of the torus. In all of these papers, the additional resultants $S^\alpha$ and $T^\alpha$ are assumed zero, which is consistent with the assumption of a membrane state, so that the thickness change cannot be prescribed on an edge of the shell. However, since the material is assumed incompressible, the thickness change which is necessary to maintain an incompressible state can be calculated throughout the shell from the incompressibility condition.

The object of the present paper is to derive for a shell a governing system of equations which can admit the prescription of a definite symmetric thickness change on the boundaries and is capable of predicting the thickness change throughout the shell, when the strains on the middle surface are finite. The material is assumed elastic, isotropic and incompressible.

While the effects of bending are included in our analysis, it is assumed that the incompressibility condition need only be satisfied on the middle surface of the shell, so that the quadratic term in (1) and $T^\alpha$ can be neglected. This assumption restricts the deformation to cases where the membrane strains are much larger than the bending strains over most of the shell. It admits only a symmetric thickness change and excludes any antisymmetric motion of the bounding surfaces of the shell with respect to its middle surface. Such antisymmetric motion would have been produced by bending strain if the incompressibility condition were enforced throughout the shell wall.

As far as the shell theory is concerned, our approach for the derivation of the governing equations parallels that used by Naghdi and Nordgren [8]. Aside of our specialization to incompressible materials and convected middle surface coordinates, the essential difference between the equations of [8] and those of this paper lies in our admission of a change in the thickness of the shell which is not admitted by the Kirchhoff hypothesis used in [8].

The basic field equations and the kinematical quantities, which are required for the admission of a thickness change, could have also been obtained from the Cosserat surface theory derived, for example, by Green *et al.* in [9]. We have not adopted this approach, because the inclusion of such concepts as incompressibility, which is a natural condition of a 3-dimensional theory, seems to be more straightforward with the approach used in [8].

Tensor notation is used throughout the paper. The quantities belonging to the deformed shell are denoted by lower case letters, while their duals in the undeformed shell are denoted by capital letters. Greek indices take the values 1 and 2, while Latin indices take the values 1, 2 and 3. Diagonally repeated indices denote summation over the range of the indices. Comma denotes partial differentiation.

## STRAIN

Let $s$ denote the middle surface of the deformed shell, and let us define a general coordinate system $x^i$ such that $x^\alpha$ are the curvilinear coordinates on the middle surface of the deformed shell and $x^3$ is the distance measured along the normal of the middle surface of the deformed shell. Let $\mathbf{a}_3(x^\alpha)$ be the unit normal vector, and let the base vectors of the $x^\alpha$ corrdinates be defined by

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} = \partial\mathbf{r}/\partial x^\alpha \tag{2}$$

where $\mathbf{r}(x^\alpha)$ is the position vector of a point lying on the deformed middle surface of the shell. The position vector of a generic point of the deformed shell space is then

$$\mathbf{r}^*(x^i) = \mathbf{r}(x^\alpha) + x^3\mathbf{a}_3(x^\alpha) \tag{3}$$

subject to the restrictions

$$\mathbf{r}^*_{,\alpha} \cdot \mathbf{a}_3 = 0, \qquad \mathbf{a}_3 \cdot \mathbf{a}_3 = 1 \tag{4}$$

A similar coordinate system is defined for the undeformed shell and is denoted by $X^A = (X^\Delta, X^3)$. Therefore, the position vector of a point belonging to the undeformed shell is given by

$$\mathbf{R}^*(X^A) = \mathbf{R}(X^\Delta) + X^3\mathbf{A}_3(X^\Delta) \tag{5}$$

together with

$$\mathbf{R}^*_{,\Delta} \cdot \mathbf{A}_3 = 0, \qquad \mathbf{A}_3 \cdot \mathbf{A}_3 = 1 \tag{6}$$

We now consider problems for which, except in localized zones, the membrane stresses are large in comparison with the bending stresses, so that, as far as the thickness change is concerned, over most of the shell the symmetric part of the transverse normal strain is the dominant factor. Thus, as a first approximation, we assume that the deformation of the shell is characterized by the mapping

$$x^\alpha = X^\alpha, \qquad x^3 = \lambda(X^\alpha)X^3 \tag{7}$$

and its inverse. We are then assuming that the middle surface coordinates are convected, and the resulting deformation is such that the normal lines to the undeformed middle surface remain normal lines to the deformed middle surface, but the surfaces originally parallel to the undeformed middle surface do not remain parallel to the deformed middle surface of the shell.

From (7), the deformation gradients are given by

$$\|x^i_{,A}\| \equiv \|\partial x^i/\partial X^A\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda_{,1}X^3 & \lambda_{,2}X^3 & \lambda \end{Vmatrix} \tag{8}$$

and hence

$$|x^i_{,A}| = \det x^i_{,A} = \lambda \tag{9}$$

Since the inverse of the mapping (7) exists, we can solve equations (8) to get

$$\|X^A_{,i}\| = \|\partial X^A/\partial x^i\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda_{,1}x^3/\lambda^2 & -\lambda_{,2}x^3/\lambda^2 & 1/\lambda \end{Vmatrix} \tag{10}$$

and hence

$$|X^A_{,i}| = \det X^A_{,i} = 1/\lambda \tag{11}$$

The square of the line elements in the deformed and undeformed shell are then given by

$$ds^2 = d\mathbf{r}^* . d\mathbf{r}^* = g_{ij}\, dx^j\, dx^j \tag{12}$$

$$dS^2 = d\mathbf{R}^* . d\mathbf{R}^* = G_{AB}\, dX^A\, dX^B = c_{ij}\, dx^j \tag{13}$$

where

$$g_{ij} = \mathbf{r}^*_{,i} . \mathbf{r}^*_{,j} \tag{14}$$

$$G_{AB} = \mathbf{R}^*_{,A} . \mathbf{R}^*_{,B} \tag{15}$$

are the space metric tensors of the deformed and undeformed shells and

$$c_{ij} = G_{AB}X^A_{,i}X^B_{,j} \tag{16}$$

is the covariant Cauchy–Green deformation tensor. Since we focus our attention to incompressible materials, we prefer the use of the Cauchy–Green deformation measures over the Almansi–Hamel strain tensor, because the constitutive law will be stated in terms of $c_{ij}$. Owing to the special shell coordinates employed in describing the deformation, the metric tensor of the deformed shell space becomes

$$g_{\alpha\beta} = \mathbf{r}^*_{,\alpha} . \mathbf{r}^*_{,\beta} = \mu^\nu_\alpha \mu^\sigma_\beta a_{\nu\sigma} \tag{17a}$$

$$g_{\alpha 3} = \mathbf{r}^*_{,\alpha} . \mathbf{a}_3 = 0 \tag{17b}$$

$$g_{33} = \mathbf{a}_3 . \mathbf{a}_3 = 1 \tag{17c}$$

where

$$a_{\nu\sigma} = \mathbf{a}_\nu . \mathbf{a}_\sigma = g_{\nu\sigma}|_{x^3=0} \tag{18}$$

is the surface metric tensor of the deformed shell and

$$\mu^\nu_\alpha = \delta^\nu_\alpha - x^3 b^\nu_\alpha \tag{19}$$

is the so-called shell tensor, $b^\nu_\alpha$ is the mixed curvature tensor, whose covariant components are given by

$$b_{\alpha\beta} = \mathbf{a}_3 . \mathbf{a}_{\alpha,\beta} \tag{20}$$

Let $\mathbf{v}$ be the displacement vector which maps the points of the undeformed shell space onto the points of the deformed shell space. Because of the special mapping assumed by (7), the displacement vector must be of the form

$$\mathbf{v} = \mathbf{r}^* - \mathbf{R}^* = v^\alpha \mathbf{a}_\alpha + v^3 \mathbf{a}_3 = \mathbf{u}(x^\alpha) + x^3 \boldsymbol{\beta}(x^\alpha) \tag{21}$$

In (21), $v^\alpha$ and $v^3$ denote the shifted components of the displacement vector and are related to the components of $\mathbf{u}$ (middle surface displacement vector) and $\boldsymbol{\beta}$ (rotation vector) by

$$v^\alpha = u^\alpha + x^3\beta^\alpha \tag{22a}$$

$$v^3 = v_3 = u_3 + x^3\beta_3 \tag{22b}$$

Since the deformations are assumed to be large, linear displacements in $x^3$ will lead to nonlinear strain expressions which is characterized by

$$c_{\alpha\beta} = {}_0c_{\alpha\beta} + x^3\ {}_1c_{\alpha\beta} + (x^3)^2\ {}_2c_{\alpha\beta} \tag{23a}$$

$$c_{\alpha 3} = {}_0c_{\alpha 3} + x^3\ {}_1c_{\alpha 3} \tag{23b}$$

$$c_{33} = {}_0c_{33} \tag{23c}$$

where, the submeasures of Cauchy–Green deformation tensor, ${}_nc_{ij}$, may be easily deduced through (10), (16) and (22):

$${}_0c_{\alpha\beta} = A_{\alpha\beta} = a_{\alpha\beta} - (\phi_{\alpha\beta} + \phi_{\beta\alpha}) + \phi^\sigma_{.\alpha}\phi_{\sigma\beta} + \phi^3_{.\alpha}\phi^3_{.\beta} \tag{24a}$$

$${}_1c_{\alpha\beta} = -2B_{\alpha\beta}/\lambda = -2b_{\alpha\beta} - (k_{\alpha\beta} + k_{\beta\alpha}) + b^\sigma_\alpha\phi_{\sigma\beta} + b^\sigma_\beta\phi_{\sigma\alpha} + k_{\sigma\beta}\phi^\sigma_{.\alpha} + k_{\sigma\alpha}\phi^\sigma_{.\beta} + k^3_\alpha\phi^3_\beta + k^3_\beta\phi^3_\alpha \tag{24b}$$

$${}_2c_{\alpha\beta} = (B_{\sigma\alpha}B^\sigma_\beta + \lambda_{,\alpha}\lambda_{,\beta}/\lambda^2)/\lambda^2 = b_{\sigma\alpha}b^\sigma_\beta + b^\sigma_\alpha k_{\sigma\beta} + b^\sigma_\beta k_{\sigma\alpha} + k^\sigma_\alpha k_{\sigma\beta} + k^3_\alpha k^3_\beta \tag{24c}$$

$${}_0c_{\alpha 3} = (\beta_3 - 1)\phi^3_{.\alpha} + (\phi^\sigma_{.\alpha} - \delta^\sigma_\alpha)\beta_\sigma = 0 \tag{24d}$$

$${}_1c_{\alpha 3} = -\lambda_{,\alpha}/\lambda^3 = (\beta_3 - 1)\beta_{3,\alpha} + \beta^\sigma_{|\alpha}\beta_\sigma \tag{24e}$$

$${}_0c_{33} = 1/\lambda^2 = (1-\beta_3)^2 + \beta^\sigma\beta_\sigma \tag{24f}$$

where, after Naghdi and Nordgren [8], we have used

$$\phi^\sigma_{.\alpha} = u^\sigma_{|\alpha} - b^\sigma_\alpha u_3, \qquad \phi^3_{.\alpha} = u_{3,\alpha} + b_{\alpha\sigma}u^\sigma \tag{25a}$$

$$k^\sigma_{.\alpha} = \beta^\sigma_{|\alpha} - b^\sigma_\alpha\beta_3, \qquad k^3_{.\alpha} = \beta_{3,\alpha} + b_{\alpha\sigma}\beta^\sigma \tag{25b}$$

In the above expressions, a stroke denotes covariant differentiation with respect to the deformed middle surface metric tensor $a_{\alpha\beta}$.

The displacement components defined by (22) are not independent, but, because of (7), they must satisfy the condition of zero transverse shear strain on the middle surface, namely ${}_0c_{\alpha 3} = 0$, which relates the rotations $\beta^\alpha$ to the middle surface displacements $u^\alpha$, $u_3$. However, the transverse shear strain away from the middle surface can be expressed through (24e) and (24f) as

$${}_1c_{\alpha 3} = \tfrac{1}{2}\,{}_0c_{33,\alpha} \tag{26}$$

and it only vanishes when the thickness change is constant with respect to $x^\alpha$. When the Kirchhoff hypothesis is fully invoked, i.e. when

$$x^3 = X^3 \tag{27a}$$

which, from (7), implies that

$$\lambda = 1 \tag{27b}$$

then we must make $\beta_3$ satisfy the two equations

$$ {}_1c_{\alpha 3} = 0 \tag{28a} $$

$$ {}_0c_{33} = 1 \tag{28b} $$

It is evident from (26) that the satisfaction of (28b) implies the satisfaction of (28a).

Stress constitutive equations of incompressible elastic materials can be expressed in terms of the contravariant Cauchy–Green deformation tensor $c^{ij}$ and its inverse $(c^{-1})^{ij}$, which are defined by the equations

$$ c^{ij} = g^{ik}g^{jl}c_{kl} \tag{29a} $$

$$ (c^{-1})^{ij}c_{jk} = \delta^i_k \tag{29b} $$

Using the definition of $c_{ij}$, we can invert equation (29b) to give

$$ (c^{-1})^{ij} = G^{AB}x^i_{,A}x^j_{,B} \tag{30} $$

Since constitutive equations of shell variables involve integration along the thickness of the deformed shell, explicit expressions of $c^{ij}$ and $(c^{-1})^{ij}$ as functions of $x^3$ are needed. To this end, we recall that the shell tensor $\mu^\nu_\alpha$ is nonsingular and hence a unique inverse exists which has the property

$$ (\mu^{-1})^\alpha_\sigma\mu^\sigma_\beta = \delta^\alpha_\beta \tag{31} $$

and, as shown by Naghdi [10], is expressible in convergent series of $x^3$ as

$$ (\mu^{-1})^\alpha_\sigma = \sum_{n=0}^{\infty} \overset{n}{b}{}^\alpha_\sigma (x^3)^n \tag{32} $$

where the coefficients $\overset{n}{b}{}^\alpha_\sigma$ are given by

$$ \overset{n}{b}{}^\alpha_\sigma = \underbrace{b^\alpha_\gamma b^\gamma_\delta \dots b^\nu_\sigma}_{n\,\text{factors}};\ \overset{n}{b}{}^\alpha_\sigma = \delta^\alpha_\sigma \tag{33} $$

Hence the contravariant metric tensor $g^{ij}$ can be expressed as

$$ g^{\alpha\beta} = (\mu^{-1})^\alpha_\sigma(\mu^{-1})^\beta_\nu a^{\sigma\nu} \tag{34a} $$

$$ g^{\alpha 3} = 0 \tag{34b} $$

$$ g^{33} = 1 \tag{34c} $$

where

$$ a^{\sigma\nu} = g^{\sigma\nu}|_{x^3=0} $$

is the contravariant surface metric tensor of the deformed shell. Similar results also hold for the undeformed shell. The series expansions of $c^{ij}$ and $(c^{-1})^{ij}$ can then easily be obtained from (29a) and (30) by using (8), (10), (32) and (34), together with their duals in the undeformed shell. The resulting expressions are

$$c^{\alpha\beta} = g^{\alpha\sigma}g^{\beta\gamma}\left[G_{\sigma\gamma}+\frac{\lambda_{,\sigma}\lambda_{,\gamma}}{\lambda^2}(X^3)^2\right]$$

$$= a^{\alpha\sigma}a^{\beta\gamma}\sum_{n=0}^{\infty}\overset{n}{(C)}_{\sigma\gamma}(x^3)^n \tag{35a}$$

$$c^{\alpha 3} = -\frac{\lambda_{,\gamma}}{\lambda^2}g^{\alpha\gamma}X^3 = -\frac{a^{\alpha\sigma}\lambda_{,\gamma}}{\lambda^3}\sum_{n=1}^{\infty} n\overset{n-1}{(b)}{}^{\gamma}_{\sigma}(x^3)^n \tag{35b}$$

$$c_{33} = \frac{1}{\lambda^2} \tag{35c}$$

$$(c^{-1})^{\alpha\beta} = G^{\alpha\beta} = A^{\alpha\sigma}\sum_{n=0}^{\infty}(n+1)\,\overset{n}{(B)}{}^{\beta}_{\sigma}\left(\frac{x^3}{\lambda}\right)^n \tag{36a}$$

$$(c^{-1})^{\alpha 3} = G^{\alpha\beta}\lambda_{,\beta}X^3 = A^{\alpha\sigma}\lambda_{,\gamma}\sum_{n=1}^{\infty}\overset{n-1}{(B)}{}^{\gamma}_{\sigma}\left(\frac{x^3}{\lambda}\right)^n \tag{36b}$$

$$(c^{-1})^{33} = G^{\alpha\beta}\lambda_{,\alpha}\lambda_{,\beta}(X^3)^2+\lambda^2 = \lambda^2$$

$$+A^{\alpha\sigma}\lambda_{,\alpha}\lambda_{,\gamma}\sum_{n=2}^{\infty}(n-1)\,\overset{n-2}{(B)}{}^{\gamma}_{\sigma}\left(\frac{x^3}{\lambda}\right)^n \tag{36c}$$

where the coefficients $\overset{n}{(C)}_{\sigma\gamma}$ are defined by

$$\overset{0}{(C)}_{\sigma\gamma} = \overset{0}{(C)}{}^{\alpha\beta}_{\sigma\gamma}A_{\alpha\beta} \tag{37a}$$

$$\overset{1}{(C)}_{\sigma\gamma} = \overset{1}{(C)}{}^{\alpha\beta}_{\sigma\gamma}A_{\alpha\beta}-\frac{2}{\lambda}\overset{0}{(C)}{}^{\alpha\beta}_{\sigma\gamma}B_{\alpha\beta} \tag{37b}$$

and for $n \geq 2$ by

$$\overset{n}{(C)}_{\sigma\gamma} = \overset{n}{(C)}{}^{\alpha\beta}_{\sigma\gamma}A_{\alpha\beta}-\frac{1}{\lambda}\overset{n-1}{(C)}{}^{\alpha\beta}_{\sigma\gamma}B_{\alpha\beta}+\frac{1}{\lambda^2}\overset{n-2}{(C)}{}^{\alpha\beta}_{\sigma\gamma}\left[B_{\alpha\delta}B^{\delta}_{\beta}+\frac{\lambda_{,\alpha}\lambda_{,\beta}}{\lambda^2}\right] \tag{37c}$$

in which

$$\overset{n}{(C)}{}^{\alpha\beta}_{\sigma\gamma} = \sum_{k=0}^{n}(k+1)(n-k+1)\overset{k}{(b)}{}^{\alpha}_{\sigma}\overset{n-k}{(b)}{}^{\beta}_{\gamma} \tag{38}$$

and $\overset{n}{(B)}{}^{\alpha}_{\sigma}$ is the dual of $\overset{n}{(b)}{}^{\alpha}_{\sigma}$ [defined by equation (33)] in the undeformed shell.

Finally, we need the relations between the thicknesses of the deformed and undeformed shells, which are denoted by $h$ and $h_0$, respectively. It follows from (7) that the thickness of the deformed shell can be expressed as

$$h(x^\alpha) = \int_{-h/2}^{h/2} dx^3 = \lambda(x^\alpha)\int_{-h_0/2}^{h_0/2} dX^3 = \lambda(x^\alpha)h_0 \tag{39}$$

The function $\lambda(x^\alpha)$, which characterizes the symmetric thickness change, can be easily expressed as a function of the displacement components by comparing the two alternative descriptions of ${}_0c_{33}$ as given by (24f), which leads to

$$\lambda(x^\alpha) = \{(1-\beta_3)^2+\beta^\sigma\beta_\sigma\}^{-1/2} \tag{40}$$

## EQUATIONS OF EQUILIBRIUM

In the absence of body forces, the stress equilibrium equations in normal coordinates are (Naghdi [10], Section 5.2)

$$(\mu\mu^\alpha_\sigma\tau^{\sigma\beta})_{|\beta}-\mu b^\alpha_\beta\tau^{3\beta}+(\mu\mu^\alpha_\sigma\tau^{\sigma 3})_{,3} = 0 \tag{41a}$$

$$(\mu\tau^{3\beta})_{|\beta}+\mu\mu^\sigma_\beta b_{\sigma\alpha}\tau^{\beta\alpha}+(\mu\tau^{33})_{,3} = 0 \tag{41b}$$

$$\varepsilon_{\sigma\gamma}\mu\mu^\sigma_\alpha\mu^\gamma_\beta\tau^{\alpha\beta} = 0 \tag{41c}$$

where $\varepsilon_{\sigma\gamma}$ is the permutation tensor of the deformed surface, and

$$\mu = \det \mu^\alpha_\sigma$$

Multiplication of (41a) and (41b) by $x^3$ and rearrangement of terms leads to

$$(\mu\mu^\alpha_\sigma x^3\tau^{\sigma\beta})_{|\beta}-\mu\tau^{\alpha 3}+(\mu\mu^\alpha_\sigma x^3\tau^{\sigma 3})_{,3} = 0 \tag{42a}$$

$$(\mu x^3\tau^{3\beta})_{|\beta}+\mu\mu^\sigma_\beta b_{\sigma\alpha}x^3\tau^{\beta\alpha}+(\mu x^3\tau^{33})_{,3}-\mu\tau^{33} = 0 \tag{42b}$$

With reference to [10], the stress resultant and couple tensors, per unit length of the coordinate curves on the deformed middle surface, can be defined as

$$\begin{Bmatrix} N^{\alpha\beta} \\ M^{\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} \mu\mu^\beta_\sigma \begin{Bmatrix} 1 \\ x^3 \end{Bmatrix} \tau^{\alpha\sigma}\,dx^3 \tag{43a}$$

$$\begin{Bmatrix} Q^{\alpha} \\ S^{\alpha} \end{Bmatrix} = \int_{-h/2}^{h/2} \mu \begin{Bmatrix} 1 \\ x^3 \end{Bmatrix} \tau^{\alpha 3}\,dx^3 \tag{43b}$$

$$N^{33} = \int_{-h/2}^{h/2} \mu\tau^{33}\,dx^3 \tag{43c}$$

where $h$ is the thickness of the deformed shell, $N^{\alpha\beta}$, $Q^\alpha$ and $M^{\alpha\beta}$ are the usual stress resultants and couples, while $S^\alpha$ and $N^{33}$, which arise in connection with the symmetric transverse normal strain effect, are the moment of the transverse shear stress and the average normal stress, respectively. The equations of equilibrium in terms of stress resultants and couples may then be obtained by integrating (41) and (42) across the thickness of the deformed shell, which leads to

$$N^{\beta\alpha}{}_{|\beta}-b^\alpha_\beta Q^\beta+l^\alpha = 0 \tag{44a}$$

$$Q^\beta{}_{|\beta}+b_{\alpha\beta}N^{\beta\alpha}+l^3 = 0 \tag{44b}$$

$$M^{\beta\alpha}{}_{|\beta}-Q^\alpha+m^\alpha = 0 \tag{44c}$$

$$S^\beta{}_{|\beta}+b_{\alpha\beta}M^{\beta\alpha}-N^{33}+m^3 = 0 \tag{44d}$$

and

$$\varepsilon_{\alpha\sigma}(N^{\sigma\alpha}-b^\sigma_\beta M^{\beta\alpha}) = 0 \tag{44e}$$

In comparison to a shell theory subjected to the Kirchhoff hypothesis, or to the theory of [2 and 3], (44d) is an additional equilibrium equation which must be satisfied if the thickness change (or $\beta_3$) is to be prescribed on an edge of the shell. This equation, however, is contained within the general system of equations derived for linear theory in [1].

The effect of the inclusion of the thickness change is also displayed in the load terms, which are obtained through the integration of (41) and (42), together with the appropriate use of the Leibnitz rule of integral calculus, since the limits of the integrals are dependent on $x^\alpha$. Thus, the load terms are given by

$$l^\alpha = |\mu\mu^\alpha_\beta\tau^{\beta 3}|^{h/2}_{-h/2} - \tfrac{1}{2}h_{,\beta}[\mu\mu^\alpha_\sigma\tau^{\sigma\beta}]^{h/2}_{-h/2} \tag{45a}$$

$$l^3 = |\mu\tau^{33}|^{h/2}_{-h/2} - \tfrac{1}{2}h_{,\beta}[\mu\tau^{3\beta}]^{h/2}_{-h/2} \tag{45b}$$

$$m^\alpha = |\mu\mu^\alpha_\sigma x^3(\tau^{\sigma 3} - \tfrac{1}{2}h_{,\beta}\tau^{\sigma\beta})|^{h/2}_{-h/2} \tag{45c}$$

$$m^3 = |\mu x^3(\tau^{33} - \tfrac{1}{2}h_{,\beta}\tau^{3\beta})|^{h/2}_{-h/2} \tag{45d}$$

where

$$[\mu\tau^{3\beta}]^{h/2}_{-h/2} = \mu\tau^{3\beta}|_{h/2} + \mu\tau^{3\beta}|_{-h/2}$$

Equations (45) reduce to the previously given expressions (Naghdi [10], Section 5.1), if we set $h_{,\beta} = 0$.

Since the thickness of the deformed shell may be variable with respect to $x^\alpha$, the normal direction of the deformed middle surface of the shell may not coincide with the normal direction of the bounding surfaces (or faces) of the shell. One immediate consequence of this is that a pressure load acting on the faces of the shell may have a component along the tangent of the deformed shell. Since for rubber-like incompressible materials a pressure load is more common than the loads in the tangential direction of the faces, we shall obtain expressions for $l^\alpha$ and $l^3$ in terms of a given pressure load.

Let us note that the equation of the upper face of the deformed shell is given by

$$\mathbf{r}^*(x^\alpha) = \mathbf{r}(x^\alpha) + \tfrac{1}{2}h(x^\alpha)\mathbf{a}_3(x^\alpha) \tag{46}$$

and let the unit outer normal and unit tangent vectors to the upper face be denoted by $\mathbf{n}$ and $\mathbf{t}_\alpha$. Using (46), we get that

$$\mathbf{n} = (\mathbf{r}^*_{,1} \times \mathbf{r}^*_{,2})/|\mathbf{r}^*_{,1} \times \mathbf{r}^*_{,2}||_{h/2} = n_i\mathbf{g}^i|_{h/2} = -\tfrac{1}{2}kh_{,\alpha}\mathbf{g}^\alpha + k\mathbf{a}_3|_{h/2} \tag{47a}$$

$$\mathbf{t}_\alpha = \mathbf{r}^*_{,\underline{\alpha}}/|\mathbf{r}^*_{,\underline{\alpha}}||_{h/2} = b_{\underline{\alpha}}\mathbf{g}_{\underline{\alpha}} + \tfrac{1}{2}b_{\underline{\alpha}}h_{,\underline{\alpha}}\mathbf{a}_3|_{h/2} \tag{47b}$$

where bars under indices are used to suspend summation,

$$\mathbf{g}^\alpha = g^{\alpha\beta}\mathbf{g}_\beta$$

$$\mathbf{g}^3 = \mathbf{a}^3$$

denote the contravariant base vectors of the shell space, $e^{\alpha\beta}$ is the Cartesian permutation symbol, and

$$g = \det g_{ij}$$

$$k = \left\{1 + \left(\frac{1}{4g}\right)e^{\alpha\beta}e^{\sigma\gamma}g_{\alpha\sigma}h_{,\beta}h_{,\gamma}\right\}^{-\frac{1}{2}}\Bigg|_{h/2} \tag{48a}$$

$$b_\alpha = \{g_{\underline{\alpha\alpha}} + h_{,\underline{\alpha}}h_{,\underline{\alpha}}/4\}^{-\frac{1}{2}}|_{h/2} \tag{48b}$$

Let

$$p^+ = p|_{h/2}, p^- = p|_{-h/2}$$

be the normal pressure on the upper and lower faces of the shell, respectively, measured per unit area of the deformed faces of the shell. The stress boundary conditions require that

$$-p^+\mathbf{n} = \mathbf{t}|_{h/2} \tag{49}$$

where

$$\mathbf{t} = \tau^{ij} n_i \mathbf{g}_j$$

is the stress vector. The vector equation (49) is equivalent to the following scalar equations, which, after some rearrangements, can be written as

$$\mu(\tau^{3\sigma} - \tfrac{1}{2}h_{,\alpha}\tau^{\alpha\sigma})|_{h/2} = \tfrac{1}{2}\mu h_{,\alpha} g^{\alpha\sigma} p|_{h/2} \tag{50a}$$

$$\mu(\tau^{33} - \tfrac{1}{2}h_{,\alpha}\tau^{\alpha 3})|_{h/2} = -\mu p|_{h/2} \tag{50b}$$

Similar expressions hold for the lower face of the deformed shell. The load terms can now be obtained directly from (45) and (50) in the form

$$l^\alpha = [\tfrac{1}{2}\mu h_{,\beta} g^{\beta\sigma} \mu^\alpha_\sigma p]^{h/2}_{-h/2} \tag{51a}$$

$$l^3 = -|\mu p|^{h/2}_{-h/2} \tag{51b}$$

It is clear that a pressure load has a component along the tangent of the deformed middle surface.

In most applications, however, $l^\alpha$ may be much smaller over most of the shell than $l^3$. To see the relative magnitudes of $l^\alpha$ and $l^3$, we choose curvilinear coordinates on the middle surface of the deformed shell such that

$$a_{\alpha\beta} = 0(1) \tag{52}$$

where 0 denotes the order of magnitude of the argument. Then the magnitudes of the other terms of (51) are given by

$$\mu^\alpha_\beta = \delta^\alpha_\beta - x^3 b^\alpha_\beta = \delta^\alpha_\beta + 0(h/r) \tag{53a}$$

$$\mu = \det \mu^\alpha_\beta = 1 + 0(h/r) \tag{53b}$$

$$g^{\alpha\beta} = 1 + 0(h/r) \tag{53c}$$

and hence, from (51), it follows that

$$l^\alpha = 0(h_{,\alpha}[p^+ + p^-]) \tag{54a}$$

$$l^3 = -(p^+ - p^-) + 0\left(\frac{h}{r}[p^+ + p^-]\right) \tag{54b}$$

where $r$ is the smallest radius of curvature of the deformed shell. It is evident that $l^\alpha$ is small compared to $l^3$ if

$$h_{,\alpha} = 0(h) \tag{55}$$

and then we may set

$$l^\alpha = 0 \tag{56a}$$

$$l^3 = -(p^+ - p^-) \tag{56b}$$

as $h \to 0$. In those situations, however, where the deformed thickness of the shell is constrained on some edges to be equal to the undeformed thickness, the contribution of pressure to $l^\alpha$, as given by (51), may be significant.

## CONSTITUTIVE EQUATIONS

For an elastic, isotropic, incompressible material, the stress constitutive equations are given by (Rivlin [11], Section 4)

$$\tau^{ij} = -p_0 g^{ij} + 2(c^{-1})^{ij} \partial \sum / \partial I_1 - 2c^{ij} \partial \sum / \partial I_2 \tag{57}$$

where

$$\sum = \sum (I_1, I_2) \tag{58}$$

denotes the strain energy function per unit undeformed volume, $p_0(x^i)$ designates the unknown hydrostatic pressure, and

$$I_1 = (c^{-1})^i_i \tag{59a}$$

$$I_2 = (c^{-1})^i_j (c^{-1})^j_i \tag{59b}$$

stand for the first and second invariants of the inverse Cauchy–Green deformation tensor. For incompressible materials, the deformation must be isochoric which means that

$$I_3 = 1 \tag{60}$$

$I_3$ being the third invariant (determinant) of the mixed inverse Cauchy–Green deformation tensor. Recalling (30), we get that

$$(c^{-1})^i_j = G^{AB} g_{kj} x^i_{,A} x^k_{,B} \tag{61}$$

and hence

$$I_3 = |(c^{-1})^i_j| = |G^{AB}||g_{kj}||x^i_{,A}|^2 = \lambda^2 g/G = 1 \tag{62}$$

where $g$ and $G$ are the determinants of the metric tensors of the deformed and undeformed shells, respectively. Because of the approximate form of the mapping (7), the incompressibility condition can be satisfied only on the middle surface of the shell, which from (62) leads to

$$(I_3)^{\frac{1}{2}} \simeq \lambda (a/A)^{\frac{1}{2}} = 1 \tag{63}$$

In (63),

$$a = |a_{\alpha\beta}| = g|_{x^3=0} \qquad A = |A_{\alpha\beta}| = G|_{x^3=0} \tag{64}$$

represents the determinants of the surface metric tensors. Since the incompressibility condition is satisfied only on the middle surface of the shell, the unknown hydrostatic pressure $p_0(x^i)$ in the constitutive relations (57) cannot vary across the thickness and hence

$$p_0(x^i) = p_0(x^\alpha) \tag{65}$$

A model which approximates the incompressible isotropic rubber-like materials has been suggested by Mooney and has a simple strain energy function of the form

$$\sum = \sum_{10} (I_1 - 3) + \sum_{01} (I_2 - 3) \tag{66}$$

where $\sum_{10}$ and $\sum_{01}$ are material constants. In the remainder of the paper, we shall confine our attention to materials with a Mooney strain energy form.

The constitutive equations for stress resultants and couples are obtained by substituting the stress constitutive equations (57) into (43) and then using (35) and (36). This process leads to an infinite series representation for the stress resultants and couples, which after truncation can be written as

$$N^{\alpha\beta} = h(N_0^{\alpha\beta}+N_2^{\alpha\beta}h^2/12)+0(h^4) \tag{67a}$$

$$M^{\alpha\beta} = M_1^{\alpha\beta}h^3/12+0(h^4) \tag{67b}$$

$$S^{\alpha} = S_1^{\alpha}h^3/12+0(h^4) \tag{67c}$$

$$N^{33} = h(N_0^{33}+N_2^{33}h^2/12)+0(h^4) \tag{67d}$$

These expressions are tensorially invariant in the sense of Naghdi [10]. The coefficients in the above equations are given by

$$N_0^{\alpha\beta} = -p_0a^{\alpha\beta}+2\sum\nolimits_{10} A^{\alpha\beta}-2\sum\nolimits_{01} a^{\alpha\gamma}a^{\beta\sigma}A_{\gamma\sigma} \tag{68a}$$

$$\begin{aligned} M_1^{\alpha\beta} = {} & -2p_0a^{\alpha\sigma}b_\sigma^\beta+4\sum\nolimits_{10} A^{\alpha\sigma}B_\sigma^\beta/\lambda-4\sum\nolimits_{01} a^{\alpha\gamma}a^{\beta\delta}(b_\gamma^\sigma A_{\sigma\delta}+b_\delta^\sigma A_{\gamma\sigma}-B_{\delta\gamma}/\lambda) \\ & -(b_\gamma^\beta+\delta_\gamma^\beta b_\sigma^\sigma)N_0^{\alpha\gamma} \end{aligned} \tag{68b}$$

$$\begin{aligned} N_2^{\alpha\beta} = {} & -3p_0a^{\alpha\gamma}b_\sigma^\beta b_\gamma^\sigma+6\sum\nolimits_{10} A^{\alpha\gamma}B_\sigma^\beta B_\gamma^\sigma/\lambda^2 \\ & -2\sum\nolimits_{01} a^{\alpha\gamma}a^{\beta\delta}\{(3b_\rho^\sigma b_\gamma^\rho A_{\sigma\delta}+4b_\delta^\rho b_\gamma^\sigma A_{\sigma\rho}+3b_\sigma^\rho b_\delta^\sigma A_{\gamma\rho}) \\ & -4(b_\gamma^\sigma B_{\sigma\delta}+b_\delta^\sigma B_{\gamma\sigma})/\lambda+(B_{\gamma\sigma}B_\delta^\sigma+\lambda_{,\gamma}\lambda_{,\delta}/\lambda^2)/\lambda^2\} \\ & -(b_\gamma^\beta+\delta_\gamma^\beta b_\sigma^\sigma)M_1^{\alpha\gamma}+(b_\sigma^\sigma b_\gamma^\beta+\tfrac{1}{2}\delta_{\rho\mu}^{\sigma\delta}b_\sigma^\rho b_\delta^\mu\delta_\gamma^\beta)N_0^{\alpha\gamma} \end{aligned} \tag{68c}$$

$$S_1^{\alpha} = (2\sum\nolimits_{10} A^{\alpha\beta}+2\sum\nolimits_{01} a^{\alpha\beta}/\lambda^2)\lambda_{,\beta}/\lambda \tag{68d}$$

$$N_0^{33} = -p_0+2\sum\nolimits_{10}\lambda^2-2\sum\nolimits_{01}/\lambda^2 \tag{68e}$$

$$N_2^{33} = 2\sum\nolimits_{10} A^{\alpha\beta}\lambda_{,\alpha}\lambda_{,\beta}/\lambda^2 \tag{68f}$$

Since the transverse shear strain on the middle surface is assumed zero, the leading term in the constitutive relation of $Q^\alpha$ is zero. Consequently, the constitutive relation of $Q^\alpha$ is not used, but $Q^\alpha$ is determined from the equilibrium equations.

In order to recover the previous results appropriate for membrane theory, we let

$$M^{\alpha\beta} = S^\alpha = Q^\alpha = 0,$$

and then the equilibrium equation (44d) yields

$$N^{33} = 0 \tag{69}$$

which serves for the determination of the unknown hydrostatic pressure $p_0$ from (68e) as

$$p_0 = 2\sum\nolimits_{10}\lambda^2-2\sum\nolimits_{01}/\lambda^2 \tag{70}$$

Elimination of $p_0$ from the constitutive relation (67a) by means of (70) now gives

$$N^{\alpha\beta} = h\{2\sum\nolimits_{10}(A^{\alpha\beta}-\lambda^2a^{\alpha\beta})-2\sum\nolimits_{01}(a^{\alpha\gamma}a^{\beta\sigma}A_{\gamma\sigma}-a^{\alpha\beta}/\lambda^2)\} \tag{71}$$

which is the constitutive relation used in membrane theory in [4–7].

It may be of interest to compare (68) to the known results in case of infinitesimal deformations of an isotropic, incompressible material. To this end, we first note that to the first order in strains

$$I_1 \simeq I_2 \simeq 3$$

so that

$$\partial \sum / \partial I_1 = \sum_{10}, \partial \sum / \partial I_2 = \sum_{01}$$

are constants. Furthermore, it can be shown that

$$2\sum_{10} + 2\sum_{01} = E/3 \tag{72}$$

where $E$ is the Young's modulus of an incompressible material. To get the appropriate constitutive coefficients valid for infinitesimal deformation we need the expression of the deformed metric tensor which to the first order in strains are

$$\begin{aligned} a_{\beta\sigma} &= A_{\beta\sigma} + 2_0 e_{\beta\sigma} \\ a^{\beta\sigma} &= A^{\beta\sigma} - A^{\beta\gamma} A^{\sigma\lambda} 2_0 e_{\gamma\lambda} \end{aligned} \tag{73}$$

where $2_0 e_{\beta\sigma}$ is the linearized middle surface strain tensor, i.e.

$$2_0 e_{\beta\sigma} = \phi_{\beta\sigma} + \phi_{\sigma\beta} \tag{74}$$

Substitution of (73) and (74) into (68a), together with the use of (72), now yields

$$N_0^{\alpha\beta} = -PA^{\alpha\beta} + (E/3)(A^{\alpha\lambda}A^{\beta\sigma} + A^{\alpha\sigma}A^{\beta\lambda})\phi_{\lambda\sigma} \tag{75}$$

where we have defined

$$P = p_0 - 2\sum_{10} + 2\sum_{01} \tag{76}$$

which is an arbitrary hydrostatic pressure.

Equation (75) checks exactly with the corresponding constitutive equation derived under the Kirchhoff hypothesis [10, § 6, equation (6.44)], if we observe that the Kirchhoff hypothesis, coupled with the incompressibility condition, requires that

$$\phi^{\lambda}_{.\lambda} = 0 \tag{77}$$

In a similar fashion, it can be shown that the remaining coefficients in (68) also reduce to their linear counterparts given in [10].

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**Абстракт**—Для больщих, упругих деформаций и несжимаемых материалов, толщина должна, в случае растяжения, изменяться. В предлагаемой работе дается теория оболочек, учитывающая заданные изменения толщины на краях оболчки, и способная определить симметрическое изменения во всей оболочке. Приводятся определяющие уравнения для несжимаемого материала, с конститутивном законом Мунея.